Lecture 17

Types of recurrences:

1. Non-linear : Hard
2. Linear : Easy

Linear Recurrences:

- Just follow the steps.

Linear Homogeneous Recurrences

Ex

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3} \quad , \quad a_0 = 7 \quad , \quad a_1 = 0 \quad , \quad a_2 = 10$$

Step 1

Move all terms to one side:

$\rightarrow \quad a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$

$\rightarrow \quad x^3 - 2x^2 - x + 2 = 0$

Side Example of characteristic eq:

$a_n - a_{n-3} = 0$

$\rightarrow x^3 - 1 = 0$

Step 2 Factorization

$$(x-2)(x-1)(x+1) = 0$$

$x_1 = 2, \; x_2 = 1, \; x_3 = -1$

Step 3 Get general solution by raising all roots to the power by n, and multiply each by different constants.

$$a_n = A(2)^n + B(1)^n + C(-1)^n$$

Step 4 Use initial conditions to set up system.

$a_0 = A(2)^0 + B(1)^0 + C(-1)^0$

$\hookrightarrow a_0 = A + B + C = 7$

$$\left.\begin{aligned} a_0 &= A + B + C = 7 \\ a_1 &= 2A + B - C = 0 \\ a_2 &= 4A + B + C = 10 \end{aligned}\right\} \text{System}$$

Step 5 Solve system:

$A = 1$

$B = 2$

$C = 4$

Step 6 Write general solution with constants plugged in to get actual solution.

Another Example

$a_n = 4a_{n-1} - 4a_{n-2}$

$a_n - 4a_{n-1} + 4a_{n-2} = 0$

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$x_1 = x_2 = 0$

so then $a_n = A2^n + Bn2^n$

Another Example

If you get $(x-2)^4$ as your roots to characteristic equation:

$a_n = A2^n + Bn2^n + Cn^2 2^n + Dn^3 2^n$

Linear Non-homogeneous Recurrences

$\boxed{Ex}$

$$f_n = \underbrace{4f_{n-1} - 3f_{n-2}}_{f_n^{(h)}} + \underbrace{4^n}_{f_n^{(p)}}$$

$$f_n = f_n^{(h)} + f_n^{(p)}$$

1. Solve for $f_n^{(h)}$

$$f_n = 4f_{n-1} - 3f_{n-2}$$

$$f_n - 4f_{n-1} - 3f_{n-2} = 0$$

$$x^2 - 4x + 3 = 0$$

$$(x-1)(x-3) = 0$$

$$x_1 = 1,\ x_2 = 3$$

$$f_n^{(h)} = A(1)^n + B(3)^n$$

2. Solve for $f_n^{(p)}$

↳ Since we have 4^n, we assume a root of the characteristic equation is $x = 4$

↳ So you just assume $f_n^{(p)} = A4^n$

↳ then plug in $f_n^{(p)}$ into f_n and set it equal to $f_n^{(p)}$

$$f_n^{(p)} = 4f_{n-1}^{(p)} - 3f_{n-2}^{(p)} + 4^n$$

$$A4^n = 4(A4^{n-1}) - 3(A4^{n-2}) + 4^n$$

Solve for A

$$A = \frac{16}{3}$$

So $f_n^{(p)} = \frac{16}{3}4^n$

So $f_n = f_n^{(a)} + f_n^{(p)}$

$f_n = A(1)^n + B(3)^n + \frac{16}{3}4^n$

$\boxed{\text{Ex}}$ $f_n = \underbrace{6f_{n-1} - 9f_{n-2}}_{f_n^{(h)}} + \underbrace{2\cdot 3^n}_{f_n^{(p)}}$

1. Solve $f_n^{(h)}$

$(x^2 - 6x + 9) = 0$

$(x-3)^2 = 0$

$x_1 = 3$, $x_2 = 3$

$f_n^{(h)} = A3^n + Bn3^n$

2. Solve for $f_n^{(p)}$

1. Notice the root is $x = 3$
2. Also notice $f_n^{(h)}$ has 2 roots equal to 3
   - ↳ from $f_n^{(p)}$ we found another root equal to 3
   - ↳ So now you know there are <u>three</u> roots equal to 3
   - ↳ so $f_n^{(p)} = Cn^2 3^n$

Now do $f_n^{(p)} = 6f_{n-1}^{(p)} - 9f_{n-2}^{(p)} + 2\cdot 3^n$

↳ from this $C = 1$

So: $\boxed{f_n = A3^n + Bn(3)^n + n^2\cdot 3^n}$